Talk at the workshop

„Macdonald polynomials"

at Clay Mathematics Institute

"Gelfand – Tsetlin bases via Laumon spaces"

03/09/2010

· Introduction

Thanks for inviting here!

· Today I will talk about Laumon spaces, first introduced by G. Laumon in 1989 (for Langlands Program).
I will give basic definitions of these spaces and then explain 2 constructions:

- the action of $Y(sl_n)$ on equiv. Borel-Moore homology of Laumon spaces and their affine analogue
- the action of quantum loop alg. $U_v(Lsl_n)$ in equiv. K-theory and affine analogue.

The constructions are quite natural and similar to those, used by H. Nakajima in his works on quiver varieties.

Laumon Spaces – Definitions

- Let C be a smooth proj. curve of genus 0, i.e. $C \simeq \mathbb{C}P^1$. Let z be a coordinate on C. There is also an action $\mathbb{C}^* \circlearrowright C$ by formula $v(z) \doteq v^{-2}z$. Then $C^{\mathbb{C}^*} = \{0, \infty\}$.

- Let W – n-dim v. space with basis $w_1, \dots, w_n$.
  $T \subset G = GL_n \subset Aut(W)$ – Cartan torus
  $\widetilde{T}$ – 2^n-fold cover of T; $\widetilde{T} \ni \underline{t} = (t_1, \dots, t_n)$ acts as $\underline{t}(w_i) = t_i^2 w_i$
  $\mathcal{B}$ – flag variety of G.
  $\underline{d} = (d_1, \dots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} \rightsquigarrow$ Laumon quasiflag space $\mathcal{Q}_{\underline{d}}$

Def: $\mathcal{Q}_{\underline{d}} :=$ moduli space of flags of loc. free subsheaves
$0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \dots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_C$, s.t.
$rk\, \mathcal{W}_k = k$, $\deg \mathcal{W}_k = -d_k$.

Fact: It is known $\mathcal{Q}_{\underline{d}}$ – smooth proj. variety of $\dim = 2(d_1 + \dots + d_{n-1}) + \dim \mathcal{B}$

We are interested in the following loc. closed subvariety $\mathcal{R}_{\underline{d}}$ inside $\mathcal{Q}_{\underline{d}}$ formed by the flags as above, s.t.
$\mathcal{W}_i \subset \mathcal{W}$ – vector subbundle in a nbhd of $\infty \in C$ and the fiber of $\mathcal{W}_i$ at ∞ equals span $\langle w_1, \dots, w_i \rangle \subset W$.

Fact: It is known $\mathcal{R}_{\underline{d}}$ – smooth quasiproj. variety of $\dim = 2(d_1 + \dots + d_{n-1})$
There is an obvious action $\widetilde{T} \times \mathbb{C}^* \circlearrowright \mathcal{R}_{\underline{d}}$. The fixed points are parametrized by $\widetilde{\underline{d}} = (d_{ij})_{i \geq j}$, s.t. $d_i = \sum_j d_{ij}$ and $d_{kj} \geq d_{ij}$ for $i \geq k \geq j$

$\mathcal{W}_1 = \mathcal{O}_C(-d_{11} \cdot 0) \cdot w_1$
$\mathcal{W}_2 = \mathcal{O}_C(-d_{21} \cdot 0) \cdot w_1 + \mathcal{O}_C(-d_{22} \cdot 0) w_2$
$\vdots$
$\mathcal{W}_{n-1} = \mathcal{O}_C(-d_{n-1,1} \cdot 0) w_1 + \dots + \mathcal{O}_C(-d_{n-1,n-1} \cdot 0) w_{n-1}$

Natural correspondences

- For $i \in \{1, \dots, n-1\}$ there is a natural correspondence

$E_{\underline{d},i} \subset Q_{\underline{d}} \times Q_{\underline{d}+i}$ formed by $(W_\bullet, W_\bullet')$, s.t. $\begin{cases} W_j = W_j', & j \neq i \\ W_i' \subset W_i \end{cases}$

In other words:

$E_{\underline{d},i}$ = moduli space of flags of loc. free sheaves

$$0 \subset W_1 \subset \dots \subset W_{i-1} \subset W_i' \subset W_i \subset W_{i+1} \subset \dots \subset W_{n-1} \subset W,$$

s.t. $\mathrm{rk}\, W_k = k$, $\deg W_k = -d_k$, $\mathrm{rk}\, W_i' = i$, $\deg W_i' = -d_i - 1$.

Fact: $E_{\underline{d},i}$ – smooth proj. alg. variety of $\dim = 2(d_1 + \dots + d_{n-1}) + \dim B + 1$.

There is a natural line bundle on $E_{\underline{d},i}$:

L_i – natural line bundle on $E_{\underline{d},i}$, whose fiber at a point $(W_\bullet, W_\bullet')$ equals $\Gamma(C, W_i/W_i')$

We also have a picture which we'll need in the future

$$\begin{tikzcd} & E_{\underline{d},i} \arrow[r, "z"', "\mathrm{supp}\, W_i/W_i'"] \arrow[dl, "p"'] \arrow[dr, "q"] & C \\ Q_{\underline{d}} & & Q_{\underline{d}+i} \end{tikzcd}$$

Rmk: One can restrict all this for $R_{\underline{d}} \times R_{\underline{d}+i}$. Then $E_{\underline{d},i}$ – smooth qproj. of $\dim = 2d_1 + \dots + 2d_{n-1} + 1$.

Example: $n = 2$.

$Q_{\underline{d}} = \{\mathcal{O}(-d) \hookrightarrow \mathcal{O} \oplus \mathcal{O}\} = \{\mathcal{O} \hookrightarrow \mathcal{O}(d) \oplus \mathcal{O}(d)\} = \Gamma(\mathbb{CP}^1, \mathcal{O}(d) \oplus \mathcal{O}(d))$

up to $\mathbb{C}^* \Rightarrow \dim Q_{\underline{d}} = 2d + 1$

Brief overview of equiv. homology

- We use Borel-Moore homology, i.e. $H_*^{BM}(X) := H_*(\overset{\text{one-point compactification}}{\hat{X}} := X \cup \{\infty\}, \infty)$
- For a Lie group G define $H_*^G(X) := H_*(EG \times_G X)$, where $EG \longrightarrow BG$ - universal G-bundle over classifying space of G.
- Properties: 1) given $f : X \xrightarrow{G\text{-equiv.}} Y$ there exists a pullback $f^* : H_*^G(Y) \to H_*^G(X)$

  2) considering $\pi : X \longrightarrow pt$ endows $H_*^G(X)$ with a structure of $H_*^G(pt)$-module.

  3) If T - n-torus $H_*^T(pt) = \mathbb{C}[x_1, \ldots, x_n]$.

  4) If $f : X \to Y$ - proper G-equiv. map $\Rightarrow \exists\, f_* : H_*^G(X) \to H_*^G(Y)$

Localization Thm (in case $G = T = (\mathbb{C}^*)^k$): If X^T is finite then the restriction map $H := H_*^T(X) \otimes_{H_*^T(pt)} \mathrm{Frac}(H_*^T(pt)) \xrightarrow{\simeq} H_*^T(X^T) \otimes_{H_*^T(pt)} \mathrm{Frac}(H_*^T(pt))$ is an isom.

Upshot of thm: The classes $[p] := i_{p*} 1$ ($i_p : \underset{\in X^T}{pt} \longrightarrow X$) form a basis of H.

Brief overview of equiv. K_0-groups

- Let X - qproj. variety / $\mathbb{C}$; ~~[illegible]~~ alg. group $G \curvearrowright X$ alg.

|| $K^G(X) :=$ Grothendiek group of the abelian category $Coh^G(X)$

- Pushforward for proper morphisms

Let X, Y - qproj. G-varieties, $f: X \longrightarrow Y$ - proper G-equiv

$\rightsquigarrow f_*: K^G(X) \longrightarrow K^G(Y)$ defined on coherent sheaves by f-las

$f_*(\mathcal{F}) := \sum (-1)^i [R^i f_* \mathcal{F}]$ for $\mathcal{F} \in Coh^G(X)$.

- There is still a Localization Thm, which is the same as in equiv. homology.

Pull-back with support - well defined.

Exterior power : For a vector bundle E we define

$\Lambda_u E := \sum_{i=0}^{rk E} u^i \Lambda^i E$, $\det E := \Lambda^{rk E} E$. They extend to

$\Lambda_u : K_0^G(X) \longrightarrow K_0^G(X)[[u]]$, $\det : K_0^G(X) \to K_0^G(X)$

Convolution 1

Let X_1, X_2, X_3 - nonsingular qproj. var., $p_{ab}: X_1 \times X_2 \times X_3 \to X_a \times X_b$.

$Z_{12} \subset X_1 \times X_2$, $Z_{23} \subset X_2 \times X_3$ - closed subvarieties and

$p_{13}: p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \longrightarrow X_1 \times X_3$ - proper.

|| $Z_{12} \circ Z_{23} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$

|| $K_{12} * K_{23} := p_{13*}(p_{12}^* K_{12} \overset{L}{\otimes}_{X_1 \times X_2 \times X_3} p_{23}^* K_{23})$ for $K_{12} \in K^G(Z_{12}), K_{23} \in K^G(Z_{23})$

Convolution 2 : Given X_1, X_2 - nonsingular, $Z_{12} \subset X_1 \times X_2$ - closed,

$Z_{12} \xrightarrow{p_2} X_2$ - proj, $\mathcal{F} \in K^G(Z_{12}) \rightsquigarrow K^G(X_1) \longrightarrow K^G(X_2)$

$$E \longmapsto p_{2*}(\mathcal{F} \otimes p_1^* E).$$

Results proved by Finkelberg and Braverman

- Thm 1 (H^T) The following operators give rise to the action of $U(gl_n)$ on $V := \bigoplus_{\underline{d}} H_*^{\widetilde{T}\times\mathbb{C}^*}(R_{\underline{d}}) \otimes_{H^{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H_*^{\widetilde{T}\times\mathbb{C}^*}(pt))$. Moreover, there is a unique isom. Ψ of $U(gl_n)$-modules V and universal Verma module B carrying $1 \in H_0^{\widetilde{T}\times\mathbb{C}^*}(R_{\underline{0}}) \subset V$ to the lowest weight vector $1 \in B$.

$$E_{ii} = \hbar^{-1}x_i + d_{i-1} - d_i + i - 1 : V_{\underline{d}} \longrightarrow V_{\underline{d}}$$
$$f_i = E_{i,i+1} = p_* q^* \quad : V_{\underline{d}} \longrightarrow V_{\underline{d}-i}$$
$$e_i = E_{i+1,i} = -q_* p^* \quad : V_{\underline{d}} \longrightarrow V_{\underline{d}+i}$$

Definition of Universal Verma module:

Let $\mathcal{U} = U(gl_n)$ - over $\mathbb{C}(\mathfrak{t}+\mathbb{C})$. Subalgebra $\mathcal{U}_{\le 0}$ - gener. by $E_{ii}, E_{i,i+1}$. It acts on $\mathbb{C}(\mathfrak{t}+\mathbb{C})$ as follows: $E_{i,i+1}$ act trivially, E_{ii} - by multiplication by $\hbar^{-1}x_i + i - 1$. $\mathcal{B} := \mathcal{U} \otimes_{\mathcal{U}_{\le 0}} \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$

Thm 2 (K^T): The following operators give rise to the action of $U_v(gl_n)$ on $M := \bigoplus_{\underline{d}} K_{(0)}^{\widetilde{T}\times\mathbb{C}^*}(R_{\underline{d}}) \otimes_{K^{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(K^{\widetilde{T}\times\mathbb{C}^*}(pt))$.

Moreover there is an isom. (unique) $\Psi : M \longrightarrow \mathfrak{M}$ - universal Verma module carrying $[\mathcal{O}_{R_{\underline{0}}}] \in M_0$ to the lowest vector $1 \in \mathfrak{M}$.

$$t_i := t_i v^{d_{i-1} - d_i + i - 1} : M_{\underline{d}} \longrightarrow M_{\underline{d}}$$
$$e_i := t_{i+1}^{-1} v^{d_{i+1} - d_i - i + 1} p_* q^* : M_{\underline{d}} \longrightarrow M_{\underline{d}-i}$$
$$f_i := -t_i^{-1} v^{d_i - d_{i-1} + i - 1} q_*(L_i \otimes p^*) : M_{\underline{d}} \longrightarrow M_{\underline{d}+i}.$$

Definition of Univ. Verma module over $U_v(gl_n)$:

$\mathfrak{M} := U_v(gl_n) \otimes_{U_v(gl_n)_{\le 0}} \mathbb{C}(\widetilde{T}\times\mathbb{C}^*)$, where $U_v(gl_n)$ - generated by $t_i^{\pm 1}, f_i$ and it acts on $\mathbb{C}(\widetilde{T}\times\mathbb{C}^*)$
f_i - acts trivially
t_i - acts by multiplication by $t_i v^{i-1}$

Gelfand–Tsetlin basis for repres. of gl_n

- Let me briefly recall what the G–T basis is.
- Let us consider $gl_1 \subset gl_2 \subset \ldots \subset gl_n$.

Let $L(\lambda)$ – finite dim. repres. of gl_n ← indexed by $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, s.t. $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$

Branching Rule: The restriction $L(\lambda)|_{gl_{n-1}}$ is isom. to the direct sum of pairwise inequivalent irr. repr $\bigoplus_\mu L'(\mu)$ summed over all μ, s.t. $\lambda_i - \mu_i \in \mathbb{Z}_+$, $\mu_i - \lambda_{i+1} \in \mathbb{Z}_+$ $\forall\ i = \overline{1, n-1}$.

Corollary: Repeating this further up to gl_1 we get (up to a rescaling) a basis of $L(\lambda)$ parametrized by G–T patterns Λ

$$\begin{matrix} \lambda_{n1} & & \lambda_{n2} \ldots & & \lambda_{nn} \\ \vee\!| & \wedge\!| & & & \\ \lambda_{n-1,1} & & \lambda_{n-1,2} \ldots & \lambda_{n-1,n-1} & \\ & \vdots & & & \\ \lambda_{21} & & \lambda_{22} & & \\ \vee\!| & & \wedge\!| & & \\ & \lambda_{11} & & & \end{matrix}$$

where the upper row coincides with λ.

$\| \; l_{ki} := \lambda_{ki} - i + 1$.

Thm: There exists a basis $\{\xi_\Lambda\}$ in L_λ, s.t. the action of gl_n is given by

$$E_{kk}\,\xi_\Lambda = \Big(\sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i}\Big)\xi_\Lambda$$

$$E_{k,k+1}\,\xi_\Lambda = -\sum_{i=1}^{k} \frac{(l_{ki} - l_{k+1,1})\cdot\ldots\cdot(l_{ki} - l_{k+1,k+1})}{(l_{ki} - l_{k1})\cdot\ldots\hat{i}\ldots\cdot(l_{ki} - l_{kk})}\,\xi_{\Lambda+\delta_{ki}}$$

$$E_{k+1,k}\,\xi_\Lambda = \sum_{i=1}^{k} \frac{(l_{ki} - l_{k-1,1})\cdot\ldots\cdot(l_{ki} - l_{k-1,k-1})}{(l_{ki} - l_{k1})\cdot\ldots\hat{i}\ldots\cdot(l_{ki} - l_{kk})}\,\xi_{\Lambda-\delta_{ki}}$$

Rmk: These formulas appeared in a short overview without any proofs. The construction of ξ_Λ via raising/lowering operators is due to Gelfand, 1962.

Connection with G-T patterns

- $(\underline{d}) = (d_{ij})_{n-1 \geq i \geq j \geq 1} \rightsquigarrow$ G-T pattern $\Lambda = \Lambda(\underline{d}) := (\lambda_{ij})_{n \geq i \geq j \geq 1}$

by $\lambda_{nj} := \hbar^{-1} x_j + j - 1 \quad (n \geq j \geq 1)$

$\lambda_{ij} := \hbar^{-1} x_j + j - 1 - d_{ij} \quad (n-1 \geq i \geq j \geq 1)$

Thm 1 ($H^{T \times \mathbb{C}^*}$): The isomorphism $\Psi : V \longrightarrow B$ takes

$[\underline{d}]$ to $(-\hbar)^{-|d|} \xi_{\Lambda(\underline{d})}$, where $|d| = d_1 + \_\_ + d_{n-1}$

Thm 2 ($K^{T \times \mathbb{C}^*}$): The isomorphism $\Psi : V \longrightarrow \mathfrak{M}$ takes

$[\underline{d}]$ to $(v^2 - 1)^{-|\underline{d}|} \prod_j t_j^{\sum_{i \geq j} d_{ij}} \, v^{\sum_i i d_i - \frac{|d|}{2} - \frac{\sum_{i,j} d_{ij}^2}{2}} \, \xi_{\Lambda(\underline{d})}$

↑

Both results are proved in

Feigin – Finkelberg – Frenkel – Rybnikov

Yangian of sl_n

$A_{n-1} = (a_{kl})_{1\le k,l\le n-1}$ - Cartan matrix of sl_n

The Yangian $Y(sl_n)$ is the free $\mathbb{C}[\hbar]$-algebra generated by $x^\pm_{k,r}$, $h_{k,r}$ $\left(\begin{smallmatrix}1\le k\le n-1\\ r\in\mathbb{N}\end{smallmatrix}\right)$:

(1) $[h_{k,r}, h_{l,s}]=0$, $[h_{k,0}, x^\pm_{l,s}] = \pm a_{kl}\, x^\pm_{l,s}$

(2) $2[h_{k,r+1}, x^\pm_{l,s}] - 2[h_{k,r}, x^\pm_{l,s+1}] = \pm\hbar a_{kl}(h_{k,r}x^\pm_{l,s} + x^\pm_{l,s}h_{k,r})$

(3) $[x^+_{k,r}, x^-_{l,s}] = \delta_{kl}\, h_{k,r+s}$

(4) $2[x^\pm_{k,r+1}, x^\pm_{l,s}] - 2[x^\pm_{k,r}, x^\pm_{l,s+1}] = \pm\hbar a_{kl}(x^\pm_{k,r}x^\pm_{l,s} + x^\pm_{l,s}x^\pm_{k,r})$

(5) $[x^\pm_{k,r},[x^\pm_{k,p},x^\pm_{l,s}]] + [x^\pm_{k,p},[x^\pm_{k,r},x^\pm_{l,s}]]=0$, $k=l\pm1$ $\forall\, p,r,s\in\mathbb{N}$.

These can be rewritten using generating functions

$h_k(u) \doteq 1+\sum_{r=0}^{\infty} h_{k,r}\hbar^{-r}u^{-r-1}$, $x^\pm_k(u)\doteq \sum_{r=0}^{\infty} x^\pm_{k,r}\hbar^{-r}u^{-r-1}$

(2') $\partial_u\partial_v\, h_k(u)x^\pm_l(v)(2u-2v\mp a_{kl}) = -\partial_u\partial_v\, x^\pm_l(v)h_k(u)(2v-2u\mp a_{kl})$

(4') $\partial_u\partial_v\, x^\pm_k(u)x^\pm_l(v)(2u-2v\mp a_{kl}) = -\partial_u\partial_v\, x^\pm_l(v)x^\pm_k(u)(2v-2u\mp a_{kl})$

Remark 1: Historically Michela Varagnolo constructed an action of $Y(Lg)$ in the equiv. homology of quiver varieties and then Nakajima constructed his action of $U_q(Lg)$ in equiv. K-theory.

Remark 2: There exists another definition of $Y(gl_n)$ using R-matrix.

Affine Yangian $\hat{Y}(sl_n)$

Affine Yangian $\hat{Y}(sl_n)$ is a $\mathbb{C}[\hbar,\hbar']$-alg. defined in the same way as $Y(\hat{sl}_n)$, except for relations (2, 4) for the pairs $(k,l)=(1,n)$ and $(n,1)$. These relations are modified as follows:

define "shifted series":

$'h_n(u) := h_n(u+\frac{\hbar'}{\hbar}-\frac{n}{2}) = 1+\sum_{r=0}^{\infty} {'h_{n,r}}\,\hbar^{-r}u^{-r-1}$,

$'x^\pm_n(u) := x^\pm_n(u+\frac{\hbar'}{\hbar}-\frac{n}{2}) = \sum_{r=0}^{\infty} {'x^\pm_{k,r}}\,\hbar^{-r}u^{-r-1}$. Now the new relations:

$2[{'h_{n,r+1}}, x^\pm_{1,s}] - 2[{'h_{n,r}}, x^\pm_{1,s+1}] = \mp\hbar({'h_{n,r}}\,x^\pm_{1,s} + x^\pm_{1,s}\,{'h_{n,r}})$

$2[h_{1,r+1}, {'x^\pm_{n,s}}] - 2[h_{1,r}, {'x^\pm_{n,s+1}}] = \mp\hbar(h_{1,r}\,{'x^\pm_{n,s}} + {'x^\pm_{n,s}}\,h_{1,r})$

$2[{'x^\pm_{n,r+1}}, x^\pm_{1,s}] - 2[{'x^\pm_{n,r}}, x^\pm_{1,s+1}] = \mp\hbar({'x^\pm_{n,r}}\,x^\pm_{1,s} + x^\pm_{1,s}\,{'x^\pm_{n,r}})$.

Quantum loop algebra

A_{n-1} – Cartan matrix of sl_n.

$U_v(Lsl_n)$ – associative $\mathbb{Q}(v)$-algebra, generated by $e_{k,r}, f_{k,r}, v^h, h_{k,m}$ $\left(\begin{smallmatrix}1\le k,l\le n-1\\ r\in\mathbb{Z},\, m\in\mathbb{Z}\setminus\{0\}\end{smallmatrix}\right)$

(1) $\psi_k^s(z)\psi_l^{s'}(w)=\psi_l^{s'}(w)\,\psi_k^s(z)$

(2) $(z-v^{\pm a_{kl}}w)\psi_l^s(z)x_k^\pm(w)=x_k^\pm(w)\psi_l^s(z)(v^{\pm a_{kl}}z-w)$

(3) $[x_k^+(z),x_l^-(w)]=\frac{\delta_{kl}}{v-v^{-1}}\{\delta(\frac{w}{z})\psi_k^+(w)-\delta(\frac{z}{w})\psi_k^-(z)\}$

(4) $(z-v^{\pm 2}w)x_k^\pm(z)x_k^\pm(w)=x_k^\pm(w)x_k^\pm(z)(v^{\pm 2}z-w)$

(5) $(z-v^{\pm a_{kl}}w)x_k^\pm(z)x_l^\pm(w)=x_l^\pm(w)x_k^\pm(z)(v^{\pm a_{kl}}z-w)$. , $k\neq l$.

(6) $\{x_i^s(z_1)x_i^s(z_2)x_{i\pm1}^s(w)-(v+v^{-1})x_i^s(z_1)x_{i\pm1}^s(w)x_i^s(z_2)+x_{i\pm1}^s(w)x_i^s(z_1)x_i^s(z_2)\}+\{z_1\leftrightarrow z_2\}=0$

where $s,s'=\pm$,

$\delta(z)\doteq\sum_{r=-\infty}^{\infty}z^r$; $x_k^+(z)\doteq\sum_{r=-\infty}^{+\infty}e_{k,r}z^{-r}$; $x_k^-(z)\doteq\sum_{r=-\infty}^{\infty}f_{k,r}z^{-r}$;

$$\psi_k^\pm(z)\doteq v^{\pm h_k}\exp\left(\pm(v-v^{-1})\sum_{m=1}^{\infty}h_{k,\pm m}z^{\mp m}\right).$$

Remark: This is a subquotient of $U_v(\widehat{sl}_n)$

Toroidal quantum algebra $\ddot{U}_v(\widehat{sl}_n)$

$\ddot{U}_v(\widehat{sl}_n)$ is an associative $\mathbb{Q}(u,v)$ algebra defined in the same way as a double quantum ~~quan~~ loop algebra $U_v'(L\widehat{sl}_n)$, except for relations (2,5). These relations are modified as follows:

define "shifted" series : $\hat{x}_n^\pm(z)\doteq x_n^\pm(zv^nu^2)$, $\hat{\psi}_n^\pm(z)\doteq\psi_n^\pm(zv^nu^2)$.

Now the new relations :

$\hat{x}_n^\pm(z)x_1^\pm(w)(z-v^{\mp1}w)=(v^{\mp1}z-w)x_1^\pm(w)\hat{x}_n^\pm(z)$

$\hat{\psi}_n^s(z)x_1^\pm(w)(z-v^{\mp1}w)=x_1^\pm(w)\hat{\psi}_n^s(z)(v^{\mp1}z-w)$

$\psi_1^s(z)\hat{x}_n^\pm(w)(z-v^{\mp1}w)=\hat{x}_n^\pm(w)\psi_1^s(z)(v^{\mp1}z-w)$

Actions of quantum loop alg. and Yangian of $\mathfrak{sl}_n$

• **Thm 1** ($H^{T\times\mathbb{C}^*}$): The following operators give rise to the action of $Y(\mathfrak{sl}_n)$ on V:

$$x^+_{k,r} := p_*(c_1(L_k\cdot v^k)^r\cdot q^*) : V_{\underline{d}} \longrightarrow V_{\underline{d}-k} \qquad x^+_k(u) := \sum_{r=0}^{\infty} x^+_{k,r}\hbar^{-r}u^{-r-1}$$

$$x^-_{k,r} := -q_*(c_1(L_k\cdot v^k)^r\cdot p^*) : V_{\underline{d}} \longrightarrow V_{\underline{d}+k} \qquad x^-_k(u) := \sum_{r=0}^{\infty} x^-_{k,r}\hbar^{-r}u^{-r-1}$$

$$h_k(u) = 1+\sum_{r=0}^{\infty} h_{k,r}\hbar^{-r}u^{-r-1} := a_k(u+\tfrac{k+1}{2})^{-1}a_k(u+\tfrac{k-1}{2})^{-1}a_{k-1}(u+\tfrac{k-1}{2})a_{k+1}(u+\tfrac{k+1}{2})$$

$$: V_{\underline{d}} \longrightarrow V_{\underline{d}}[[u^{-1}]].$$

$$a_m(u) := u^m + \sum_{r=1}^{m}(-\hbar)^{-r}\left(c_r^{(r)}(\underline{W}_m) - \hbar c_r^{(r-1)}(\underline{W}_m)\right)u^{m-r}, \text{ where}$$

$c_j(\underline{W}_i) = c_j^{(j)}(\underline{W}_i)\otimes 1 + c_j^{(j-1)}(\underline{W}_i)\otimes\tau$, $\tau = [\mathcal{O}(1)] \in H^{\mathbb{C}^*}_2(C)$, $\underline{W}_i$ on $R_{\underline{d}}\times C$.
tautological i-dim vector bundle

Idea of proof (FFNR): Consider operators $A_m(u)$, $B_m(u)$, $C_m(u)$ from Molev's book; define $x^+_k(u) := B_k(u+\frac{k-1}{2})A_k(u+\frac{k-1}{2})^{-1}$,

$$x^-_k(u) := A_k(u+\tfrac{k-1}{2})^{-1}C_k(u+\tfrac{k-1}{2}),\quad h_k(u) = \frac{A_{k-1}(u+\frac{k-1}{2})A_{k+1}(u+\frac{k+1}{2})}{A_k(u+\frac{k-1}{2})A_k(u+\frac{k+1}{2})}$$

The formulas for the action of A_m, B_m, C_m in GT known [Molev]

Thm 2 ($K^{T\times\mathbb{C}^*}$): The following operators give rise to the action of $U_v(L\mathfrak{sl}_n)$ on M:

$$e_{k,r} := t_{k+1}^{-1}v^{d_{k+1}-d_k+1-k}p_*((L_k v^k)^{\otimes r}\otimes q^*) : M_{\underline{d}} \longrightarrow M_{\underline{d}-k}$$

$$f_{k,r} := -t_k^{-1}v^{d_k-d_{k-1}-1+k}q_*(L_k\otimes(L_k v^k)^{\otimes r}\otimes p^*) : M_{\underline{d}} \longrightarrow M_{\underline{d}+k}$$

$$x^+_k(z) := \sum_{r=-\infty}^{\infty} e_{k,r}z^{-r},\quad x^-_k(z) := \sum_{r=-\infty}^{\infty} f_{k,r}z^{-r}$$

$$\psi^{\pm}_k(z)\Big|_{M_{\underline{d}}} = t_{k+1}^{-1}t_k v^{d_{k+1}-2d_k+d_{k-1}-1}\left(\frac{b_{k-1}(zv^{-k})b_{k+1}(zv^{-k-2})}{b_k(zv^{-k})b_k(zv^{-k-2})}\right)^{\pm} \in M_{\underline{d}}[[z^{\mp 1}]]$$

where $b_m(z) := 1+\sum_{1\le j\le m}\left(\Lambda^j_{(j)}(\underline{W}_m) - v\Lambda^j_{(j-1)}(\underline{W}_m)\right)(-z)^{-j}$,

$$\Lambda^j\underline{W}_m =: \Lambda^j_{(j)}(\underline{W}_m)\otimes 1 + \Lambda^j_{(j-1)}(\underline{W}_m)\otimes[\mathcal{O}(1)]$$

Proof: Straightforward.

Affine analogue of Laumon Spaces

- Let X be another copy of $\mathbb{C}P^1$ w/ coordinate y, $\mathbb{C}^* \circlearrowright X$ $c(x)=c^{-2}x$

$S := C \times X$, $\mathcal{D}_\infty := C \times \infty_X \cup \infty_C \times X$, $\mathcal{D}_0 := C \times 0_X$

Given an n-tuple $\underline{d} = (d_0, \dots, d_{n-1}) \in \mathbb{Z}^n_{\geq 0} \rightsquigarrow$ Parabolic sheaf of deg $\underline{d}$.

Def: Parabolic sheaf $\mathcal{F}_\bullet$ of degree $\underline{d}$ – is an infinite flag of torsion free coherent sheaves of $rk = n$ on S: $\ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$, s.t.

(a) $\mathcal{F}_{k+n} = \mathcal{F}_k(\mathcal{D}_0)$ $\forall k$

(b) $ch_1(\mathcal{F}_k) = k[\mathcal{D}_0]$ $\forall k$.

(c) $ch_2(\mathcal{F}_k) = d_i$ for $i \equiv k \pmod n$

(d) $\mathcal{F}_0$ – loc. free at $\mathcal{D}_\infty$ and trivialized at $\mathcal{D}_\infty$: $\mathcal{F}_0|_{\mathcal{D}_\infty} = W \otimes \mathcal{O}_{\mathcal{D}_\infty}$.

(e) $\forall -n \leq k \leq 0$ $\mathcal{F}_k$ is loc. free at $\mathcal{D}_\infty$.

and $\mathcal{F}_k/\mathcal{F}_{-n}$, $\mathcal{F}_0/\mathcal{F}_k$ (both supported at $\mathcal{D}_0 = C \times 0_X \subset S$) are both loc. free at pt. $\infty_C \times 0_X$. Moreover the local sections of $\mathcal{F}_k|_{\infty_C \times X}$ are those sections of $\mathcal{F}_0|_{\infty_C \times X} = W \otimes \mathcal{O}_X$, which take value in $\langle w_1, \dots, w_{n-k} \rangle \subset W$ at $0_X \in X$.

Correspondences

$E_{\underline{d},i} \subset \mathcal{P}_{\underline{d}} \times \mathcal{P}_{\underline{d}+i}$ – formed by pairs $(\mathcal{F}_\bullet, \mathcal{F}_\bullet')$, s.t. $\begin{array}{l} \mathcal{F}_j = \mathcal{F}_j' \ (j \not\equiv i) \\ \mathcal{F}_j' \subset \mathcal{F}_j \ (j \equiv i) \end{array}$

Fact: $E_{\underline{d},i}$ – smooth qproj. alg. variety of $\dim = 2\sum_{i=0}^{n-1} d_i + 1$

Line bundle: Each $E_{\underline{d},i}$ is equipped with a natural line bundle L_j $(j \equiv i)$ whose fiber at $(\mathcal{F}_\bullet, \mathcal{F}_\bullet')$ equals $\Gamma(C, \mathcal{F}_j/\mathcal{F}_j')$

$\boxed{\text{Fixed points – 2 approaches}}$

<u>1^{st} approach</u> So we have an action $\tilde{T}\times\mathbb{C}^*\times\mathbb{C}^* \circlearrowright \mathcal{P}_{\underline{d}}$. The number of fixed points is finite and they are parametrized by a collection of Young diagrams $\lambda=(\lambda^{kl})_{1\le k,l\le n}$, s.t.

$$\lambda^{11}\subset\lambda^{21}\subset\ldots\subset\lambda^{n1}\tilde{\subset}\lambda^{11},\ \lambda^{22}\subset\lambda^{32}\subset\ldots\subset\lambda^{12}\tilde{\subset}\lambda^{22},\ \ldots,\ \lambda^{nn}\subset\lambda^{1n}\subset\ldots\subset\lambda^{n-1,n}\tilde{\subset}\lambda^{n,n}$$

(here $\lambda\supset\mu$ if $\lambda_i\ge\mu_i\ \forall i\ge 0$, $\lambda\tilde{\supset}\mu$ if $\lambda_i\ge\mu_{i+1}\ \forall i\ge 0$) and s.t.

$$\underline{d}=\underline{d}(\lambda):=(d_0(\lambda)=d_n(\lambda),\ d_1(\lambda),\ldots,d_{n-1}(\lambda)),\quad d_j(\lambda)=\sum_{l=1}^{n}|\lambda^{jl}|$$

The corresponding parabolic sheaf $\mathcal{F}_\bullet=\mathcal{F}_\bullet(\lambda)$ is given by f-la:

$$\mathcal{F}_{k-n}=\bigoplus_{1\le l\le k} J_{\lambda^{kl}}w_l\oplus\bigoplus_{k<l\le n} J_{\lambda^{kl}}(-D_0)w_l \quad \text{for } 1\le k\le n.$$

<u>2^{nd} approach (due to Biswas)</u>

Let $\sigma: C\times X\longrightarrow C\times X$, $\sigma(z,y)=(z,y^n)$. Let $G=\mathbb{Z}/n\mathbb{Z}$.

We have an action $G\circlearrowright C\times X$, $k\cdot(x,y)=(x,\sqrt[n]{1}^{\,k}y)$.

<u>Rmk</u>: Parabolic sheaf is completely determined by $\mathcal{F}_0(-D_0)\subset\mathcal{F}_{1-n}\subset\ldots\subset\mathcal{F}_0$ satisfying (a–e)

<u>Observation</u>: $\mathcal{F}_\bullet \xleftrightarrow{1-1}$ G-invar. sheaf $\tilde{\mathcal{F}}$ on $C\times X$ + some conditions

$$\mathcal{F}^\bullet\longmapsto\tilde{\mathcal{F}}:=\sigma^*\mathcal{F}_{1-n}+\sigma^*\mathcal{F}_{2-n}(-D_0)+\ldots+\sigma^*\mathcal{F}_0((1-n)D_0)$$

<u>Rmk</u>: If $\mathcal{F}_\bullet\in\mathcal{P}_{\underline{d}}^{\tilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}$ then $\tilde{\mathcal{F}}=\bigoplus_{l=1}^{n}J_{\lambda^l}(-(l-1)D_0)w_l$, where $(\lambda^1,\ldots,\lambda^n)$ – collection of partitions given by $\lambda^l_{ni-n\lfloor\frac{k-l}{n}\rfloor+k-l}=\lambda^{kl}_i$

- For $j\in\mathbb{Z}$ let $(j \bmod n)\in\{1,\ldots,n\}$. For $i\ge j\in\mathbb{Z}$ if we denote $d_{ij}:=\lambda^{j \bmod n}_{i-j}$ we obtain a collection $(d_{ij})=\tilde{\underline{d}}=\tilde{\underline{d}}(\lambda)$ of nonnegative integers, s.t. $d_{kj}\ge d_{ij}\ (\forall i\ge k\ge j)$, $d_{i+n,j+n}=d_{ij}\ (\forall i\ge j)$, $d_{ij}=0\ (i-j\gg 0)$. $\Big\}(*)$

For $1\le k\le n$ let us write $d_k(\tilde{\underline{d}})=\sum_{j\le k}d_{kj}=\sum_{l=1}^{n}\sum_{i\le\lfloor\frac{k-l}{n}\rfloor}d_{k(l+ni)}=\sum_{l=1}^{n}\sum_{i\ge 0}\lambda^l_{ni-n\lfloor\frac{k-l}{n}\rfloor+k-l}$
$=\sum_{l=1}^{n}\sum_{i\ge 0}\lambda^{kl}_i=d_k(\lambda)$

<u>Thm</u>: The correspondence $\lambda\longmapsto\tilde{\underline{d}}(\lambda)$ is a bijection between the set of collections λ from 1^{st} approach and collections $\tilde{\underline{d}}$ satisf. $(*)$. Also: $\underline{d}(\lambda)=\underline{d}(\tilde{\underline{d}}(\lambda))$

Parabolic sheaves

- So we have $\sigma: C\times X \longrightarrow C\times X$ $\quad \sigma(z,y)=(z,y^n)$, $\Gamma = \mathbb{Z}/n\mathbb{Z}$.

Γ acts on $C\times X$ by multiplying the coordinate on X with $\sqrt[n]{1}$, i.e. generator $\gamma\in\Gamma$ acts via multiplication by $\exp(\frac{2\pi i}{n})$.

So everything is completely determined by the flag of sheaves
$$\mathcal{F}_0(-D_0)\subset \mathcal{F}_{-n+1}\subset \ldots \subset \mathcal{F}_0$$
satisfying conditions a)–e).

For $-n<k\leq 0$ consider a subsheaf $\tilde{\mathcal{F}}_k \subset \sigma^*\mathcal{F}_k$ defined as follows:
- away from $C\times\infty_X$ it coincides with $\sigma^*\mathcal{F}_k$
- the local sections of $\tilde{\mathcal{F}}_k|_{C\times\infty_X}$ are those sections of $\sigma^*\mathcal{F}_k|_{C\times\infty_X} = W\otimes\mathcal{O}_{C\times\infty_X}$ which take value at W^{k+n}, where $W^1=\langle w_1,\ldots,w_n\rangle \supset \ldots \supset W^n=\langle w_n\rangle$

$\mathcal{F}_\bullet \rightsquigarrow \Gamma$-equiv. torsion free sheaf $\tilde{\mathcal{F}}$ on $C\times X$:
$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{-n+1} + \tilde{\mathcal{F}}_{-n+2}(C\times\infty_X - C\times 0_X) + \ldots + \tilde{\mathcal{F}}_0((n-1)(C\times\infty_X - C\times 0_X))$$

Rmk: $\tilde{\mathcal{F}}|_{C\times\infty_X} \equiv W\otimes\mathcal{O}_{C\times\infty_X}$, $\tilde{\mathcal{F}}|_{\infty_C\times X}$ – trivial vector bundle $\Rightarrow$

$\Rightarrow$ its trivialization on $C\times\infty_X$ extends to a triv. on D_∞ canonically

The inverse isomorphism takes a Γ-equiv. torsion free sheaf $\tilde{\mathcal{F}}$ to

$\tilde{\mathcal{F}} \longmapsto$ the flag $\mathcal{F}_0(-D_0)\subset\mathcal{F}_{-n+1}\subset\ldots\subset\mathcal{F}_0$ where for $-n<k\leq 0$ we set $\mathcal{F}_k := \sigma_*(\tilde{\mathcal{F}}\otimes\mathcal{O}_S(kD_0))^\Gamma$.

Rmk: Let $\mathcal{M}_{n,d}$ – Gieseker moduli space of torsion free sheaves on $C\times X$ of rank n and second Chern class $d(=d_0+\ldots+d_{n-1})$, trivialized at D_∞.
Then $\tilde{\mathcal{F}}\in\mathcal{M}_{n,d}$. There is an action of Γ on W $\quad\gamma(w_l)=\exp(\frac{2\pi i l}{n})w_l$, $l=\overline{1,n}$
The action of Γ on $C\times X$ together with its action on the trivialization at D_∞ gives $\Gamma \circlearrowright \mathcal{M}_{n,d}$. We have $\tilde{\mathcal{F}}\in\mathcal{M}^\Gamma_{n,d}$. It turns out the above map $\mathcal{P}_{\underline{d}} \longrightarrow \mathcal{M}^\Gamma_{n,d}$ – isom on connected component and connected components of $\mathcal{M}^\Gamma_{n,d}$ are numbered by partitions of $d=d_0+\ldots+d_{n-1}$

$\boxed{\text{Quiver description of Laumon Spaces}}$ (follows Strømme work on Gr)

- According to Nakajima $\mathcal{M}_{n,d}$ has the following GIT description
Set $V = \mathbb{C}^d \rightsquigarrow M := \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$.

Set $M \supset \mu^{-1}(0) := \{(A, B, p, q) : AB - BA + pq = 0\}$. we define
$\mu^{-1}(0)^s$ – an open subset of stable quadruples, i.e. there is no proper subspace $V' \subset V$ stable under A, B and containing $p(W)$.
There is also $GL(V) \circlearrowright M$ preserving $\mu^{-1}(0)$ and its action on $\mu^{-1}(0)^s$ is free and $\mathcal{M}_{n,d} = \mu^{-1}(0)^s / GL(V)$ – GIT quotient.

In those terms the action of Γ is follows $\gamma(A, B, p, q) = (A, \sqrt{1} \cdot B, \sqrt{1} \cdot p, q)$
Hence the connected component of the fixed point set $\mathcal{P}_{\underline{d}} \simeq \mathcal{M}^{\Gamma}_{n,\underline{d}}$ admits the following quiver description:
choose the action of Γ on V, s.t. the χ_l- isotypic component V_l has dimension d_l $(l \in \mathbb{Z}/n\mathbb{Z})$. Then

$$\mathbf{M}^{\Gamma}_{\underline{d}} = \{(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/n\mathbb{Z}}\} = \bigoplus_{l \in \mathbb{Z}/n\mathbb{Z}} \mathrm{End}(V_l) \oplus \bigoplus_{-''-} \mathrm{Hom}(V_l, V_{l+1}) \oplus \bigoplus_{-''-} \mathrm{Hom}(W_{l-1}, V_l)$$
$$\oplus \bigoplus_{-''-} \mathrm{Hom}(V_l, W_l)$$

$$\begin{tikzcd}
{} \arrow[r, "B_{-3}"] & V_{-2} \arrow[loop above, "A_{-2}"] \arrow[r, "B_{-2}"] \arrow[d, "q_{-2}"'] & V_{-1} \arrow[loop above, "A_{-1}"] \arrow[r, "B_{-1}"] \arrow[d, "q_{-1}"'] & V_0 \arrow[loop above, "A_0"] \arrow[r, "B_0"] \arrow[d, "q_0"] & V_1 \arrow[loop above, "A_1"] \arrow[r, "B_1"] \arrow[d, "q_1"] & V_2 \arrow[loop above, "A_2"] \arrow[r, "B_2"] \arrow[d, "q_2"] & \cdots \\
& W_{-2} \arrow[ur, "p_{-1}"] & W_{-1} \arrow[ur, "p_0"] & W_0 \arrow[ur, "p_1"] & W_1 \arrow[ur, "p_2"] & W_2 \arrow[ur] &
\end{tikzcd}$$

here $\dim V_l = d_l$
$\leftarrow$ $\langle w_{l \bmod n} \rangle = W_l$ $\dim W_l = 1$.

Furthermore $\mu^{-1}(0)^{\Gamma}_{\underline{d}} = \{(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/n\mathbb{Z}} : A_{l+1} B_l - B_l A_l + p_{l+1} q_l = 0 \ \forall l\}$.
$\mu^{-1}(0)^{s,\Gamma}_{\underline{d}} = \{(A_l, B_l, p_l, q_l)_{-''-} \in \mu^{-1}(0)^{\Gamma}_{\underline{d}}$, s.t. there is no proper $\mathbb{Z}/n\mathbb{Z}$ - graded subspace $V'_\bullet \subset V_\bullet$ stable under $A_\bullet, B_\bullet$ and containing $p(W_\bullet)\}$
Also $\prod_{l \in \mathbb{Z}/n\mathbb{Z}} GL(V_l) \circlearrowright M^{\Gamma}_{\underline{d}}$ preserving $\mu^{-1}(0)^{\Gamma}_{\underline{d}}$, the action on $\mu^{-1}(0)^{s,\Gamma}_{\underline{d}}$ - free

and $\mathcal{M}_{n,\underline{d}} = \mu^{-1}(0)^{s,\Gamma}_{\underline{d}} / \prod_{l \in \mathbb{Z}/n\mathbb{Z}} GL(V_l)$

The actions of affine yangian $\widehat{Y}(\mathfrak{sl}_n)$ and quant. toroidal alg $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$

Thm 1 ($H^{\tilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}$): The following operators give rise to the action of $\widehat{Y}$ on M:

$\underline{W}_{mi}$ – the quotient of tautological vector bundles on $\mathcal{P}_{\underline{d}}\times C$ equal to $\underline{\mathcal{F}}_i/\underline{\mathcal{F}}_m$.

$$x_k^{\pm}(u)=\sum_{r=0}^{\infty} x_{k,r}^{\pm}\hbar^{-r}u^{-r-1}: M_{\underline{d}} \longrightarrow M_{\underline{d}\mp k}[[u^{-1}]]$$

$$h_k(u)=1+\sum_{r=0}^{\infty} h_{k,r}\hbar^{-r}u^{-r-1}: M_{\underline{d}}\longrightarrow M_{\underline{d}}, \text{ where}$$

$$h_i(u)=a_{mi}\left(u+\tfrac{i-1}{2}\right)^{-1}a_{mi}\left(u+\tfrac{i+1}{2}\right)^{-1}a_{m,i-1}\left(u+\tfrac{i-1}{2}\right)a_{m,i+1}\left(u+\tfrac{i+1}{2}\right)$$

$$\left(a_{mi}:=u^{i-m}+\sum_{r=0}^{\infty}(-\hbar)^{-r}\left(c_r^{(r)}(\underline{W}_{mi})-\hbar c_r^{(r-1)}(\underline{W}_{mi})\right)u^{i-m-r}\right)$$

$$x_{k,r}^{+}=p_*(c_1(L_k\cdot v^k)^r\cdot q^*);\ x_{k,r}^{-}:=-q_*(c_1(L_k\cdot v^k)^r\cdot p^*)$$

Key argument (FFNR): Uses some reduction to a stack $\mathfrak{Z}_N$, coming from usual Laumon spaces but for arbitrary big parameters

Thm 2 ($K^{\tilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}$): The following operators give rise to the action of $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ on V:

$$\psi_k^{\pm}(z)=\sum_{r=0}^{\pm\infty}\psi_{k,r}^{\pm}z^{\mp r}=t_{i+1}^{-1}t_i v^{d_{i+1}-2d_i+d_{i-1}-1}\left(\frac{b_{m,i-1}(zv^{-i})\,b_{m,i+1}(zv^{-i-2})}{b_{m,i}(zv^{-i})\,b_{m,i}(zv^{-i-2})}\right)^{\pm}:$$

$$: V_{\underline{d}}\longrightarrow V_{\underline{d}}[[z^{\mp 1}]].$$

$$x_k^{+}(z)=\sum_{r=-\infty}^{\infty}e_{k,r}z^{-r}: V_{\underline{d}}\longrightarrow V_{\underline{d}-k}[[z,z^{-1}]]$$

$$x_k^{-}(z)=\sum_{r=-\infty}^{\infty}f_{k,r}z^{-r}: V_{\underline{d}}\longrightarrow V_{\underline{d}+k}[[z,z^{-1}]]$$

$$e_{k,r}:=t_{k+1}^{-1}v^{d_{k+1}-d_k+1-k}\,p_*((L_k v^k)^{\otimes r}\otimes q^*): V_{\underline{d}}\to V_{\underline{d}-k}$$

$$f_{k,r}:=-t_k^{-1}v^{d_k-d_{k-1}-1+k}\,q_*(L_k\otimes(L_k v^k)^{\otimes r}\otimes p^*): V_{\underline{d}}\longrightarrow V_{\underline{d}+k}$$

$$b_{mi}(z):=1+\sum_{j=1}^{\infty}\left(\Lambda^j_{(j)}(\underline{W}_{mi})-v\,\Lambda^j_{(j-1)}(\underline{W}_{mi})\right)(-z)^{-j},$$ which is independent of $m<i$.

Proof: The formulas in the fixed points basis are essentially the same.

Specialization of G-T basis

- Fix a positive integer K (level) and n-tuple $\mu = (\mu_{1-n}, \dots, \mu_0) \in \mathbb{Z}^n$, s.t. $\mu_0 + K \geq \mu_{1-n} \geq \mu_{2-n} \geq \dots \geq \mu_{-1} \geq \mu_0$

<u>Rmk</u>: We view μ as a dominant weight of $\widehat{\mathfrak{gl}}_n$ of level K.

We extend μ to $\tilde{\mu} = (\tilde{\mu}_i)_{i \in \mathbb{Z}}$ setting $\tilde{\mu}_i = \mu_i + \lfloor \frac{-i}{n} \rfloor K$

$D(\mu) :=$ subset of the set D of all collections $\underline{d}$ satisfying $(*)$, (i.e. $\underline{d} = (d_{ij})_{i \geq j}$ of nonnegative integers, s.t. $d_{kj} \geq d_{ij}$ $(\forall i \geq k \geq j)$, $d_{i+n, j+n} = d_{ij}$ $(\forall i \geq j)$, $d_{ij} = 0$ $(i-j \gg 0)$) s.t. $\underline{d} \in D(\mu)$ iff $d_{ij} - \tilde{\mu}_j \leq d_{i+l, j+l} - \tilde{\mu}_{j+l}$ $\forall j \leq i, l \geq 0$

We call $D(\mu)$ "affine Gelfand-Tsetlin pattern".

Now we specialize $u := v^{-K-n}$, $t_j := v^{\tilde{\mu}_j - j + 1}$ and normalize fixed points basis $\langle \underline{d} \rangle := C_{\underline{d}}^{-1} \cdot [\underline{d}]$, where $C_{\underline{d}} = \prod_{w \in T_{\underline{d}} \mathcal{P}_d} (1-w)$ of the $T \times \mathbb{C}^* \times \mathbb{C}^*$-weights in tangent space to $\mathcal{P}_{\underline{d}}$ at pnt. $\underline{d}$.

We define $V(\mu) := \mathbb{C}(v)$-linear span of the vectors $\langle \underline{d} \rangle$ for $\underline{d} \in D(\mu)$

<u>Thm</u>: Under the above specialization we get the action of $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)/(u - v^{-K-n})$ in $V(\mu)$

<u>Sketch of proof</u>: We need to check 2 statements:

1. $\forall \underline{d} \in D(\mu)$ the denominators of $e_{i,r}\langle \underline{d}, \underline{d}' \rangle$, $f_{i,r}\langle \underline{d}, \underline{d} \rangle$ don't vanish.
2. $\forall \underline{d} \in D(\mu)$, $\underline{d}' \notin D(\mu)$ the numerators of $e_{i,r}\langle \underline{d}, \underline{d}' \rangle$, $f_{i,r}\langle \underline{d}, \underline{d}' \rangle$ vanish

Restricting $V(\mu)$ to "horizontal" $U_v(\widehat{\mathfrak{sl}}_n) \subset \ddot{U}_v(\widehat{\mathfrak{sl}}_n)$ we obtain the same named $U_v(\widehat{\mathfrak{sl}}_n)$-module with the G-T basis parametrized by $D(\mu)$

We call the fixed points basis - G-T affine basis.

<u>Conjecture</u>: $\ddot{U}_v(\widehat{\mathfrak{sl}}_n)/(u - v^{-K-n})$-module $V(\mu)$ is isomorphic to Uglov-Takemura module.

Rmk: The "horizontal" $U_v(\widehat{\mathfrak{sl}}_n) \subset \ddot{U}_v(\mathfrak{sl}_n)$ is a subalgebra generated by $\{e_{i,0}, f_{i,0}, v^{\pm h_i}\}_{1 \le i \le n}$, which is isomorphic to $U_v(\widehat{\mathfrak{sl}}_n)$.

There is also a "vertical" $U_v(\widehat{\mathfrak{sl}}_n) \subset \ddot{U}_v(\mathfrak{sl}_n)$

Rmk: This also answered positively prof. Tingley's question on crystal of cylindric plane partitions model, i.e.

"Can one lift crystal structures to get repr. of $U_q(\widehat{\mathfrak{sl}}_n)$? In particular, do cylindric plane partitions parametrize a basis for a repr. of $U_q(\widehat{\mathfrak{sl}}_n)$ in any natural way".

Rmk: In his work Denis Uglov and Takemura construct a representation of quantum toroidal alg. of type $\mathfrak{sl}_n$ on every integrable irr. highest weight module of the quantum affine algebras of type $\mathfrak{gl}_n$.